Abstract Interpretation

Jan Midtgaard

Winter School, Day 1

`http://janmidtgaard.dk/aiws15/`

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What is this course about?

Crudely simplified the history of program analysis (or static analysis) can be split in two:

- an American school of program analysis
- a French school of program analysis

The former school has its roots in *data-flow analysis* and has given rise to many important results, e.g., within optimizing compilers.

Some of you may be familiar with data-flow analysis if you've taken a compiler course.

This course is concerned with the alternative, French approach.

What is abstract interpretation?

- It is a theory of *semantics-based program analysis*
- It was initially conceived in the late 1970's by Patrick and Radhia Cousot
- It has been refined over the last 40 years
  - to new applications
  - to new kinds of semantics
  - to new programming paradigms
  - by new abstract domains
  - ...

Which is the right approach?

None of them is right or wrong — it is simply an alternative view — an eye opener to a new world.

Why? To develop new techniques, to explain existing ones, to extend or strengthen them, to formalize them.

By Friday afternoon, you will be in a position to make an informed opinion.

It is not just an academic theory: it has been used to check/verify flight control software for both Airbus and Mars missions. By the end of this course, we will read papers about those.

It'll get hairy: there will be mathematics and semantics

You take the red pill. . .

You take the red pill. . .

“. . . you stay in Wonderland and I show you how deep the rabbit-hole goes. . . ”

Learning outcomes and competences

On Friday afternoon, the participants should be able to:

- *describe* and *explain* basic analyses in terms of classical abstract interpretation.
- *apply* and *reason* about Galois connections.
- *implement* abstract interpreters on the basis of the derived program analyses.

In some sense: *thinking tools*

Like O*-analysis* is a tool for reasoning about execution time (and space), *abstract interpretation* is a tool for reasoning about analyses and properties.

Pedagogical choices / Contract

Lectures – typically mornings, sometimes with a few exercises in class

Reading – study research papers and slides (@home)

Exercises – typically afternoons, both mathematics and programming. Over the week they build up a

Project – a chance for you to apply your newly acquired skills – feel free to go crazy. . .

Your background

I'm assuming you all know about lexing, parsing, context-free grammars, abstract syntax trees (ASTs) and syntax-directed definitions/translations as taught in an undergraduate compiler course.

How many of you have taken

- □ a compiler course,

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- □ a formal semantics course?

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How many of you are

- □ students? BSc,MSc,PhD?

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- □ a compiler course,
- □ a functional programming course,
- □ a formal semantics course?

How many of you are

- □ students? BSc,MSc,PhD?
- □ developers? @JetBrains?

My background

Researcher in programming languages (abstract interpretation, semantics, functional programming)

PhD in Computer Science, Aarhus University (2007).
Since then:

- INRIA Rennes,
- Roskilde University,
- Aarhus University,
- Technical University of Denmark

I developed and ran this course in Aarhus 2010–2012

Fred Mesnard has since used it at Université de la Reunion in France

Outline

- What and how of the winter school
- Transition systems
- Math: Posets, CPOs, complete lattices, Galois connections, fixed points
- Abstract interpretation basics
- OCaml intro

Transition systems

Transition systems

Definition. *A transition system is a triple (quadruple) $\langle S, S_i, S_f, \rightarrow\rangle$ where*

- S *is a set of states*
- $S_i \subseteq S$ *is a set of initial states*
- $S_f \subseteq S$ *is an optional set of final states* $(\forall s \in S_f, s' \in S : s \not\rightarrow s')$
- $\rightarrow \subseteq S \times S$ *is a transition relation relating a state to its (possible) successors*

Example 1: Euclid's algorithm

Given two numbers $x, y \in \mathbb{N}$ we can describe Euclid's GCD algorithm as a transition system:

$$
\begin{aligned}
S &= \mathbb{N} \times \mathbb{N} \\
S_i &= \{\langle x,\, y\rangle\} \\
S_f &= \{\langle n,\, n\rangle \mid n \in \mathbb{N}\} \\
\rightarrow : \langle n,\, m\rangle &\rightarrow \langle n - m,\, m\rangle && \text{if } \; n > m \\
\langle n,\, m\rangle &\rightarrow \langle n,\, m - n\rangle && \text{if } \; n < m
\end{aligned}
$$

where we have written the transition relation using *infix notation*.
We can write it out more formally as:

$$
\begin{aligned}
\rightarrow = &\{(\langle n,\, m\rangle, \langle n - m,\, m\rangle) \mid n > m\} \\
&\cup \{(\langle n,\, m\rangle, \langle n,\, m - n\rangle) \mid n < m\}
\end{aligned}
$$

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$$S = \mathbb{N} \times \mathbb{N}$$

$$S_i = \{\langle x,\, y \rangle\}$$ ←this is an "input-specific trans.sys."

$$S_f = \{\langle n,\, n \rangle \mid n \in \mathbb{N}\}$$

$$\begin{aligned} \rightarrow :\ \langle n,\, m \rangle &\rightarrow \langle n - m,\, m \rangle && \text{if } n > m \\ \langle n,\, m \rangle &\rightarrow \langle n,\, m - n \rangle && \text{if } n < m \end{aligned}$$

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\end{aligned}
$$

Example 1: Euclid's algorithm

Given two numbers $x, y \in \mathbb{N}$ we can describe Euclid's GCD algorithm as a transition system:

$$
\begin{aligned}
S &= \mathbb{N} \times \mathbb{N} \\
S_i &= S \quad \leftarrow \text{whereas this describes all possible inputs} \\
S_f &= \{\langle n, n\rangle \mid n \in \mathbb{N}\} \\
\rightarrow &: \langle n, m\rangle \rightarrow \langle n-m, m\rangle \qquad \text{if } n > m \\
&\quad\; \langle n, m\rangle \rightarrow \langle n, m-n\rangle \qquad \text{if } n < m
\end{aligned}
$$

where we have written the transition relation using *infix notation*.
We can write it out more formally as:

$$
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\rightarrow = \{(\langle n, m\rangle, \langle n-m, m\rangle) \mid n > m\} \\
\cup \{(\langle n, m\rangle, \langle n, m-n\rangle) \mid n < m\}
\end{aligned}
$$

Example 2: Modeling a program

Modeling the program

```
x := 0;
while (x < 100) {
    x := x + 1;
}
```

as a transition system:

$$S = \mathbb{Z}$$
$$S_i = \{0\}$$
$$\rightarrow = \{(x, x') \mid x < 100 \ \wedge \ x' = x + 1\}$$

How to get from a program to a transition system is the topic of the next lecture.

For now we assume that we can model the semantics (the meaning) of a program as a transition system.

Mathematical foundations

Partially ordered sets

Definition. *A partially ordered set (poset)* $\langle S; \sqsubseteq \rangle$ *is a set* S *equipped with a binary relation* $\sqsubseteq \subseteq S \times S$ *with the following properties:*

- *Reflexive:* $\forall a \in S : a \sqsubseteq a$
- *Antisymmetric:* $\forall a, b \in S : a \sqsubseteq b \ \wedge \ b \sqsubseteq a \implies a = b$
- *Transitive:* $\forall a, b, c \in S : a \sqsubseteq b \ \wedge \ b \sqsubseteq c \implies a \sqsubseteq c$

Example 1: $\langle \mathbb{N}; \leq \rangle$ is a poset

Example 2: $\langle \wp(S); \subseteq \rangle$ is a poset

Note: $\wp(S)$ is sometimes written 2^S

Example 3: If $\langle P; \sqsubseteq \rangle$ is a poset, then $\langle P; \sqsupseteq \rangle$ is a poset

Upper and lower bounds

Let $\langle P; \sqsubseteq \rangle$ be a partially ordered set.

Definition. $u \in P$ *is an* upper bound *of* $S \subseteq P$ *iff* $\forall s \in S : s \sqsubseteq u$

Definition. $l \in P$ *is an* lower bound *of* $S \subseteq P$ *iff* $\forall s \in S : l \sqsubseteq s$

Definition. $u \in P$ *is a* least upper bound *(lub) of* $S \subseteq P$ *iff it is an upper bound of* S *and it is less than all other upper bounds:* $\forall u' \in P : (\forall s \in S : s \sqsubseteq u') \implies u \sqsubseteq u'$

Definition. $l \in P$ *is a* greatest lower bound *(glb) of* $S \subseteq P$ *iff it is an lower bound of* S *and it is greater than all other lower bounds:*
$\forall l' \in P : (\forall s \in S : l' \sqsubseteq s) \implies l' \sqsubseteq l$

Complete Partial Orders (CPOs)

Definition. *A complete partial order is a poset such that all increasing chains* $c_i, i \in \mathbb{N}$ *(*$\forall i \in \mathbb{N} : c_i \sqsubseteq c_{i+1}$*) have a least upper bound:*

$$\bigsqcup_{i \in \mathbb{N}} c_i$$

Non-example: $\langle \mathbb{N}; \leq \rangle$ is *not* a CPO. Why?

Example: $\langle \wp(S); \subseteq \rangle$ is a CPO.

Complete lattices

Definition. *A complete lattice is a poset* $\langle C; \sqsubseteq, \bot, \top, \sqcup, \sqcap \rangle$ *such that*

- *the least upper bound* $\sqcup S$ *('join') and*
- *the greatest lower bound* $\sqcap S$ *('meet') exists for every subset* S *of* C.
- $\bot = \sqcap C$ *('bottom') denotes the infimum of* C *and*
- $\top = \sqcup C$ *('top') denotes the supremum of* C.

Example 1: $\langle \wp(S); \subseteq, \emptyset, S, \cup, \cap \rangle$ is a complete lattice.

Example 2: The integers (extended with $-\infty$ and $+\infty$) is a complete lattice $\langle \mathbb{Z} \cup \{-\infty, +\infty\}; \leq, -\infty, +\infty, \max, \min \rangle$.

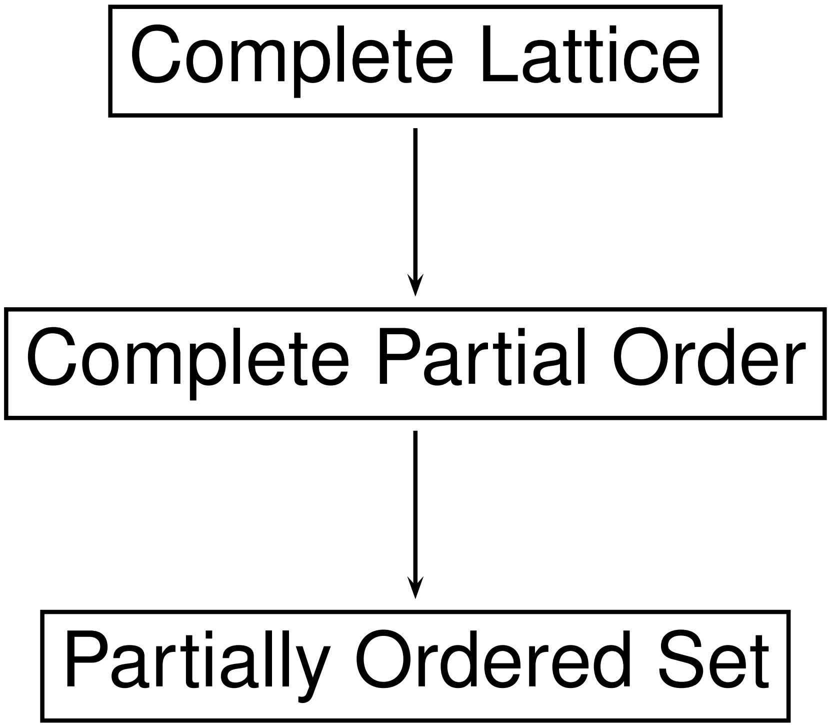
Example: A complete lattice of functions

Theorem. *The set of total functions* $D \to C$*, whose codomain is a complete lattice* $\langle C; \sqsubseteq, \bot, \top, \sqcup, \sqcap \rangle$*, is itself a complete lattice* $\langle D \to C; \dot{\sqsubseteq}, \dot{\bot}, \dot{\top}, \dot{\sqcup}, \dot{\sqcap} \rangle$ *under the pointwise ordering* $f \dot{\sqsubseteq} f' \iff \forall x. f(x) \sqsubseteq f'(x)$*, and with*

- $\dot{\bot} = \lambda x. \bot$
- $\dot{\top} = \lambda x. \top$
- $f \dot{\sqcup} g = \lambda x. f(x) \sqcup g(x)$
- $f \dot{\sqcap} g = \lambda x. f(x) \sqcap g(x)$

Here $\lambda x. \ldots$ is a mathematical function with argument x.

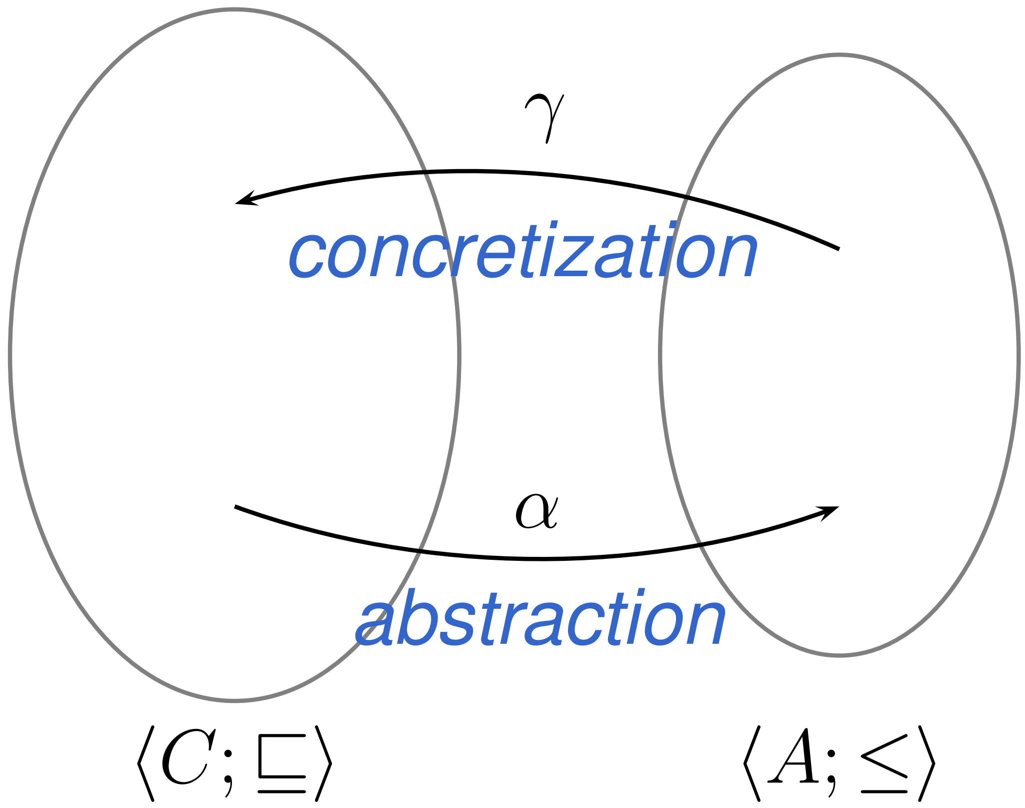
A quick comparison



Galois connections

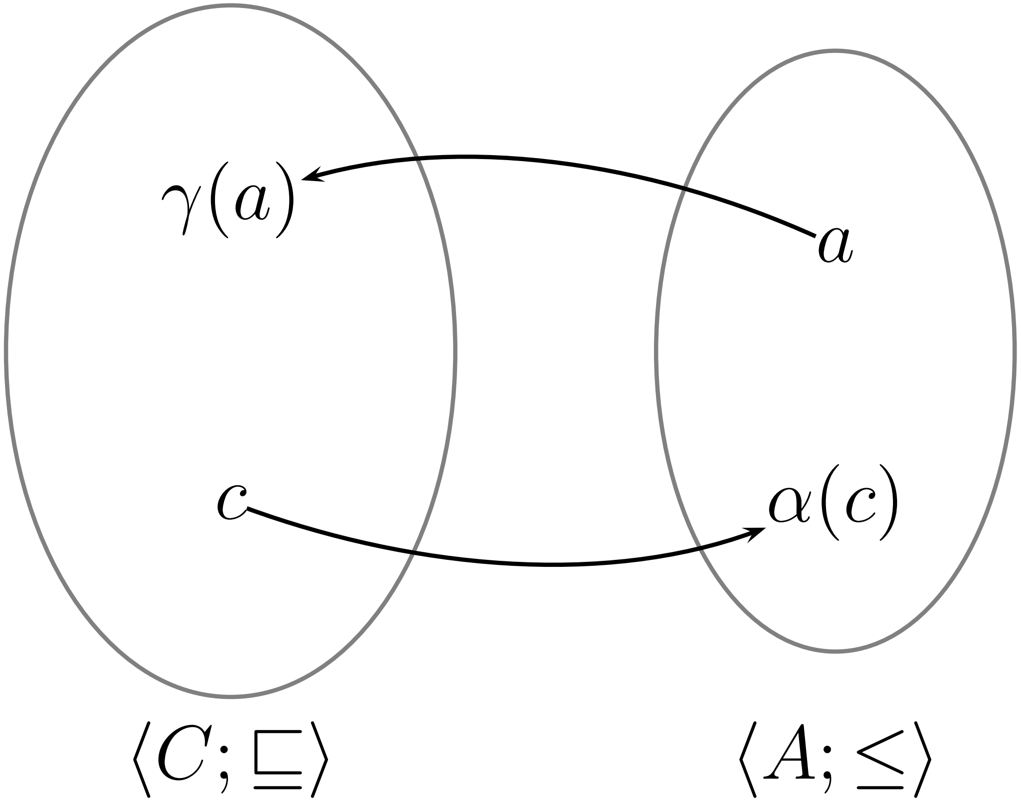
Galois connections

Definition. *A Galois connection is a pair of functions* α*,* γ *between two partially ordered sets:*



Galois connections

Definition. *A Galois connection is a pair of functions* α, γ *between two partially ordered sets:*



such that: $\forall a \in A, c \in C : \alpha(c) \leq a \iff c \sqsubseteq \gamma(a)$

Galois connections: A familiar example

You already know the pattern of moving from one side of an inequation to another from high school:

$$\forall x, y, z \in \mathbb{Z} \ : \ x + z \leq y \iff x \leq y - z$$

which we can write with α and γ as:

$$\forall x, y, z \in \mathbb{Z} \ : \ \alpha(x) \leq y \iff x \leq \gamma(y)$$

$$\text{where } \alpha(n) = n + z$$

$$\gamma(n) = n - z$$

Galois connections: An equivalent definition

Definition. *A Galois connection is a pair of functions* α *and* γ *satisfying*

(a) α *and* γ *are monotone (read: order-preserving)*
(for all $c, c' \in C : c \sqsubseteq c' \implies \alpha(c) \leq \alpha(c')$ *and*
for all $a, a' \in A : a \leq a' \implies \gamma(a) \sqsubseteq \gamma(a')$*),*

(b) $\alpha \circ \gamma$ *is reductive (for all* $a \in A : \alpha \circ \gamma(a) \leq a$*),*

(c) $\gamma \circ \alpha$ *is extensive (for all* $c \in C : c \sqsubseteq \gamma \circ \alpha(c)$*).*

Galois connections are typeset as $\langle C; \sqsubseteq \rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle A; \leq \rangle$.

Galois connection properties (1/3)

Theorem. *For a Galois connection between two complete lattices* $\langle C; \sqsubseteq, \bot_c, \top_c, \sqcup, \sqcap\rangle$ *and* $\langle A; \leq, \bot_a, \top_a, \vee, \wedge\rangle$*,* α *is a complete join-morphism (CJM):*

$$\text{for all } S_c \subseteq C : \alpha(\sqcup S_c) = \vee\alpha(S_c) = \vee\{\alpha(c) \mid c \in S_c\}$$

and γ *is a complete meet morphism (CMM):*

$$\text{for all } S_a \subseteq A : \gamma(\wedge S_a) = \sqcap\gamma(S_a) = \sqcap\{\gamma(a) \mid a \in S_a\}$$

Again: we can view these as algebraic rewriting rules.

Galois connection properties (2/3)

Theorem. *The composition of two Galois connections* $\langle C;\sqsubseteq\rangle \underset{\alpha_1}{\overset{\gamma_1}{\leftrightarrows}} \langle B;\subseteq\rangle$ *and* $\langle B;\subseteq\rangle \underset{\alpha_2}{\overset{\gamma_2}{\leftrightarrows}} \langle A;\leq\rangle$ *is itself a Galois connection:*

$$\langle C;\sqsubseteq\rangle \underset{\alpha_2\circ\alpha_1}{\overset{\gamma_1\circ\gamma_2}{\leftrightarrows}} \langle A;\leq\rangle$$

We can typeset this theorem as an inference rule:

$$\frac{\langle C;\sqsubseteq\rangle \underset{\alpha_1}{\overset{\gamma_1}{\leftrightarrows}} \langle B;\subseteq\rangle \qquad \langle B;\subseteq\rangle \underset{\alpha_2}{\overset{\gamma_2}{\leftrightarrows}} \langle A;\leq\rangle}{\langle C;\sqsubseteq\rangle \underset{\alpha_2\circ\alpha_1}{\overset{\gamma_1\circ\gamma_2}{\leftrightarrows}} \langle A;\leq\rangle}$$

Hence Galois connections stack up like Lego bricks!

Galois connection properties (3/3)

Galois connections in which α is surjective / onto (or equivalently γ is injective) are typeset as:

$$\langle C; \sqsubseteq \rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle A; \leq \rangle$$

and sometimes called Galois surjections (or insertions)

Galois connections in which α is injective / one-to-one (or equivalently γ is surjective) are typeset as:

$$\langle C; \sqsubseteq \rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle A; \leq \rangle$$

and sometimes called Galois injections

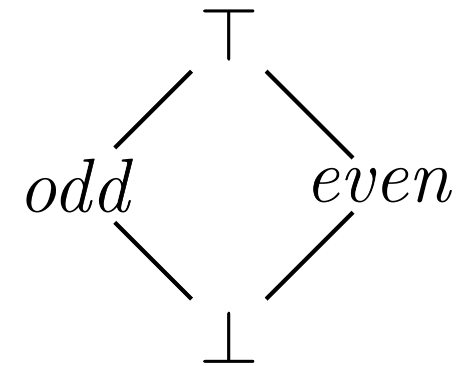
When both α and γ are surjective, the two domains are isomorphic, typeset as $\langle C; \sqsubseteq \rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle A; \leq \rangle$

Example: The Parity abstract domain

Galois connections capture *property extraction* which is essential for static analysis. Consider an abstraction into a Parity domain:

$$\langle \wp(\mathbb{N}_0); \subseteq \rangle \xleftarrow[\alpha]{\gamma} \langle Par; \sqsubseteq \rangle$$

$Par :$



(The above *Hasse diagram* defines the Parity ordering $\bot \sqsubseteq odd \sqsubseteq \top$ and $\bot \sqsubseteq even \sqsubseteq \top$)

The abstraction and concretization functions are:

$$\begin{aligned} \gamma(\bot) &= \emptyset \\ \gamma(odd) &= \{n \in \mathbb{N}_0 \mid n \bmod 2 = 1\} \\ \gamma(even) &= \{n \in \mathbb{N}_0 \mid n \bmod 2 = 0\} \\ \gamma(\top) &= \mathbb{N}_0 \end{aligned}$$

$$\alpha(N) = \begin{cases} \bot & \text{if } N = \emptyset \\ odd & \text{if } \forall n \in N : n \bmod 2 = 1 \\ even & \text{if } \forall n \in N : n \bmod 2 = 0 \\ \top & \text{otherwise} \end{cases}$$

Example: an isomorphism

Since Galois connections is a generalization of isomorphisms, they also fit nicely into the theory.

For example, we can represent a set of pairs as a function that maps a first component to its second components:

$$\langle \wp(A \times B); \subseteq \rangle \underset{\alpha}{\overset{\gamma}{\twoheadleftarrow\!\!\!\!\twoheadrightarrow}} \langle A \to \wp(B); \dot{\subseteq} \rangle$$

where $\alpha(R) = \lambda a.\{b \mid (a, b) \in R\}$

$\gamma(F) = \{(a, b) \mid b \in F(a)\}$

Fixed points

Fixed points, briefly

Definition. *a* fixed point *of a function f, is a point x such that $f(x) = x$*

Assume $f : P \to P$ operates over a poset $\langle P; \sqsubseteq \rangle$

Definition. *a* pre-fixed point *is a point x such that* $x \sqsubseteq f(x)$

Definition. *a* post-fixed point *is a point x such that* $f(x) \sqsubseteq x$

Definition. *a* least fixed point (lfp) *is a fixed point l such that for all other fixed points $l' : (f(l') = l') \implies l \sqsubseteq l'$*

Definition. *a* greatest fixed point (gfp) *is a fixed point l such that for all other fixed points* $l' : (f(l') = l') \implies l' \sqsubseteq l$

Tarski's fixed point theorem

Theorem. *If L is a complete lattice and $f : L \to L$ is a monotone function, f's fixed points themselves form a complete lattice.*

Hence Tarski tells us that *there exists a least fixed point* (and a greatest fixed point).

Abstract interpretation basics

Abstract interpretation basics

Canonical abstract interpretation approximates the *collecting semantics* of a transition system.

A standard example of a collecting semantics is the *reachable states* from a given set of initial states S_i. Given a transition function F defined as:

$$F(\Sigma) = S_i \cup \{\sigma \mid \exists \sigma' \in \Sigma : \sigma' \to \sigma\}$$

we can express the reachable states of F as the least fixed point $\operatorname{lfp} F$ of F.
For a fixed point $F(\Sigma) = \Sigma$ of F:

$$S_i \subseteq \Sigma \quad \wedge \quad \forall \sigma' \in \Sigma : \sigma' \to \sigma \implies \sigma \in \Sigma$$

which expresses the transitive closure of the states reachable from S_i.

Abstract interpretation basics

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Given a transition function F defined as:

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we can express the reachable states of F as the least fixed point $\operatorname{lfp} F$ of F.
We can obtain $\operatorname{lfp} F$ by Kleene iteration[1]:

$$\emptyset, F(\emptyset), F^2(\emptyset), F^3(\emptyset), \ldots$$

[1] In general we can only obtain $\operatorname{lfp} f$ this way if f is continuous $f(\sqcup S) = \sqcup f(S)$

Example: Collecting semantics

Program statement	State (on entry)
`x := 1;`	$\{\}$
`while (x < 100) {`	$\{\}$
`   x := x + 1;`	$\{\}$
`}`	$\{\}$

Example: Collecting semantics

Program statement	State (on entry)
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`   x := x + 1;`	$\{1\}$
`}`	$\{\}$

Example: Collecting semantics

Program statement	State (on entry)
`x := 1;`	$\{0\}$
`while (x < 100) {`	$\{1, 2\}$
`  x := x + 1;`	$\{1\}$
`}`	$\{\}$

Example: Collecting semantics

Program statement	State (on entry)
`x := 1;`	$\{0\}$
`while (x < 100) {`	$\{1, 2\}$
`  x := x + 1;`	$\{1, 2\}$
`}`	$\{\}$

Example: Collecting semantics

Program statement	State (on entry)
`x := 1;`	$\{0\}$
`while (x < 100) {`	$\{1, 2, 3\}$
`  x := x + 1;`	$\{1, 2\}$
`}`	$\{\}$

Example: Collecting semantics

Program statement	State (on entry)
`x := 1;`	$\{0\}$
`while (x < 100) {`	$\{1, 2, 3\}$
`  x := x + 1;`	$\{1, 2, 3\}$
`}`	$\{\}$

Example: Collecting semantics

Program statement	State (on entry)
`x := 1;`	$\{0\}$
`while (x < 100) {`	$\{1, 2, 3, \ldots, 98, 99\}$
`  x := x + 1;`	$\{1, 2, 3, \ldots, 98\}$
`}`	$\{\}$

Jumping forward in time. . .

Example: Collecting semantics

Program statement	State (on entry)
`x := 1;`	$\{0\}$
`while (x < 100) {`	$\{1, 2, 3, \dots, 98, 99\}$
`  x := x + 1;`	$\{1, 2, 3, \dots, 98, 99\}$
`}`	$\{\}$

Jumping forward in time. . .

Example: Collecting semantics

Program statement	State (on entry)
`x := 1;`	$\{0\}$
`while (x < 100) {`	$\{1, 2, 3, \ldots, 98, 99, 100\}$
`  x := x + 1;`	$\{1, 2, 3, \ldots, 98, 99\}$
`}`	$\{\}$

Example: Collecting semantics

Program statement	State (on entry)
`x := 1;`	$\{0\}$
`while (x < 100) {`	$\{1, 2, 3, \ldots, 98, 99, 100\}$
`   x := x + 1;`	$\{1, 2, 3, \ldots, 98, 99\}$
`}`	$\{100\}$

Example: Collecting semantics

Program statement	State (on entry)
`x := 1;`	$\{0\}$
`while (x < 100) {`	$\{1, 2, 3, \ldots, 98, 99, 100\}$
`  x := x + 1;`	$\{1, 2, 3, \ldots, 98, 99\}$
`}`	$\{100\}$

Fixed point

The strength of the collecting semantics

- The collecting semantics is ideal, i.e., it is the *most precise analysis*.
- Unfortunately it is in general uncomputable:
  an implementation is not guaranteed to terminate
- We therefore approximate the collecting semantics, by computing a fixed point over an alternative and perhaps simpler domain: an *abstract* interpretation

Abstraction and analysis using Galois connections

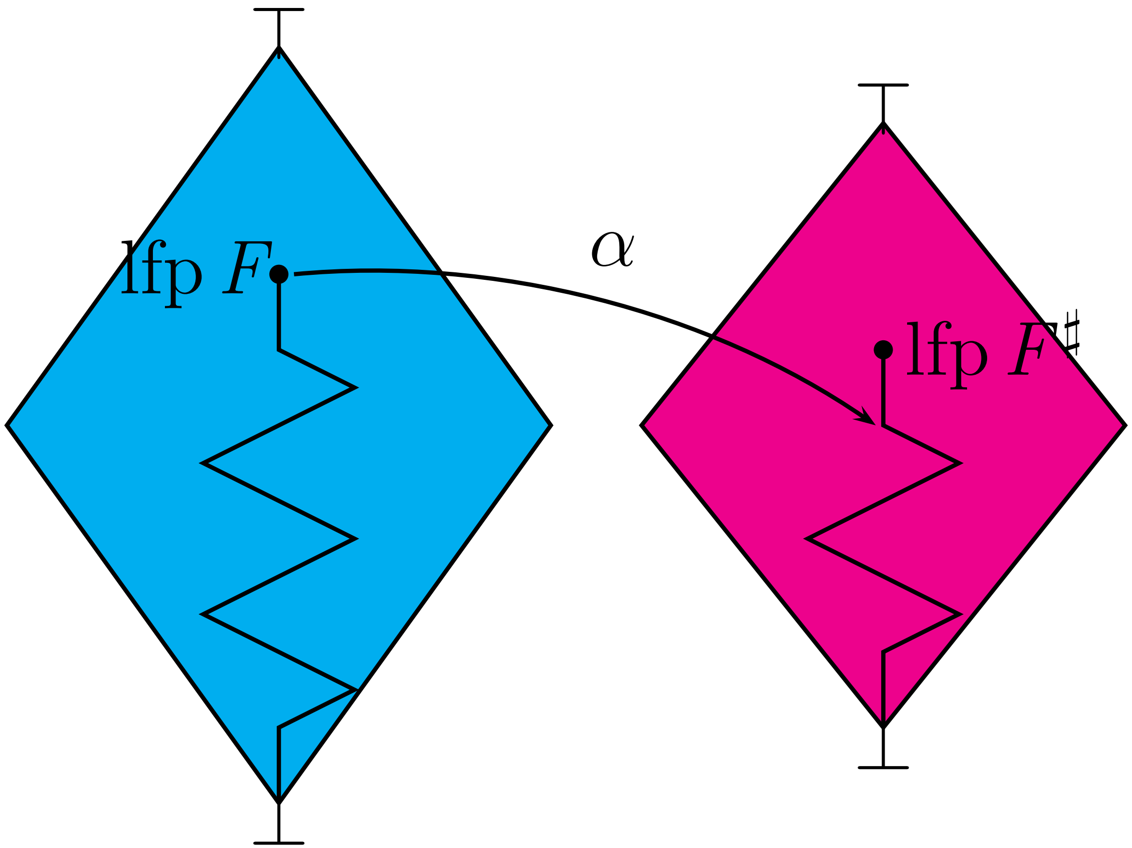
Abstractions are represented as Galois connections which connect complete lattices through α and γ.

We can derive an analysis systematically by composing the transition function with these functions: $\alpha \circ F \circ \gamma$ and gradually refine the collecting semantics into a computable analysis function by mere calculation.

Hence instead of *inventing* a static analysis, we arrive at one by a *structured abstraction* of the set of states $\wp(S)$.

Galois connection-based analysis

By the *fixed point transfer theorem* we can compute a sound approximation of the collecting semantics:



Theorem. *Let* $\langle C; \sqsubseteq \rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle A; \leq \rangle$ *be a Galois connection between complete lattices. If* F *and* $F^\sharp$ *are monotone and* $\alpha \circ F \circ \gamma \,\dot{\leq}\, F^\sharp$ *then* $\alpha(\mathrm{lfp}\, F) \leq \mathrm{lfp}\, F^\sharp$

Example: Parity analysis

Program statement	Approx. state (on entry)
`x := 1;`	$\bot$
`while (x < 100) {`	$\bot$
`x := x + 1;`	$\bot$
`}`	$\bot$

Example: Parity analysis

Program statement	Approx. state (on entry)
`x := 1;`	$even$
`while (x < 100) {`	$\bot$
`  x := x + 1;`	$\bot$
`}`	$\bot$

Example: Parity analysis

Program statement	Approx. state (on entry)
`x := 1;`	$even$
`while (x < 100) {`	odd
`  x := x + 1;`	$\bot$
`}`	$\bot$

Example: Parity analysis

Program statement	Approx. state (on entry)
`x := 1;`	$even$
`while (x < 100) {`	odd
`  x := x + 1;`	odd
`}`	odd

Example: Parity analysis

Program statement	Approx. state (on entry)
`x := 1;`	$even$
`while (x < 100) {`	$\top$
`  x := x + 1;`	odd
`}`	odd

Example: Parity analysis

Program statement	Approx. state (on entry)
`x := 1;`	$even$
`while (x < 100) {`	$\top$
`  x := x + 1;`	$\top$
`}`	$\top$

Example: Parity analysis

Program statement	Approx. state (on entry)
`x := 1;`	$even$
`while (x < 100) {`	$\top$
`  x := x + 1;`	$\top$
`}`	$\top$

Fixed point

Example: Parity analysis

Program statement	Approx. state (on entry)
`x := 1;`	$even$
`while (x < 100) {`	$\top$
`  x := x + 1;`	$\top$
`}`	$\top$

Fixed point

The result is sound: it accounts for all possible concrete executions (albeit not very precisely. . .)

Example: Parity analysis

Program statement	Approx. state (on entry)
`x := 1;`	$even \sqsupseteq \alpha(\{0\})$
`while (x < 100) {`	$\top \sqsupseteq \alpha(\{1, \ldots, 99, 100\})$
`  x := x + 1;`	$\top \sqsupseteq \alpha(\{1, \ldots, 99\})$
`}`	$\top \sqsupseteq \alpha(\{100\})$

Fixed point

The result is sound: it accounts for all possible concrete executions (albeit not very precisely. . .)

Example: Parity analysis

Program statement	Approx. state (on entry)
`x := 1;`	$\gamma(even) \supseteq \{0\}$
`while (x < 100) {`	$\gamma(\top) \supseteq \{1, \ldots, 99, 100\}$
`  x := x + 1;`	$\gamma(\top) \supseteq \{1, \ldots, 99\}$
`}`	$\gamma(\top) \supseteq \{100\}$

Fixed point

The result is sound: it accounts for all possible concrete executions (albeit not very precisely. . .)

Variations

An alternative approach

Rather than simplifying the abstract domains into finite ones, *widening* and *narrowing* permits infinite ones.

A first widening iteration overshoots the least fixed point but still ensures termination.

A second narrowing iteration improves the results of the widening iteration.

Example: Interval analysis without widening

Program statement	Approx. state (on entry)
`x := 1;`	$\bot$
`while (x < 100) {`	$\bot$
`  x := x + 1;`	$\bot$
`}`	$\bot$

Example: Interval analysis without widening

Program statement	Approx. state (on entry)
`x := 1;`	$[0; 0]$
`while (x < 100) {`	$\bot$
`  x := x + 1;`	$\bot$
`}`	$\bot$

Example: Interval analysis without widening

Program statement	Approx. state (on entry)
`x := 1;`	$[0; 0]$
`while (x < 100) {`	$[1; 1]$
`  x := x + 1;`	$\bot$
`}`	$\bot$

Example: Interval analysis without widening

Program statement	Approx. state (on entry)
`x := 1;`	$[0;0]$
`while (x < 100) {`	$[1;1]$
`  x := x + 1;`	$[1;1]$
`}`	$\perp$

Example: Interval analysis without widening

Program statement	Approx. state (on entry)
`x := 1;`	$[0; 0]$
`while (x < 100) {`	$[1; 2]$
`  x := x + 1;`	$[1; 1]$
`}`	$\bot$

Example: Interval analysis without widening

Program statement	Approx. state (on entry)
`x := 1;`	$[0; 0]$
`while (x < 100) {`	$[1; 2]$
`  x := x + 1;`	$[1; 2]$
`}`	$\bot$

Example: Interval analysis without widening

Program statement	Approx. state (on entry)
`x := 1;`	$[0;0]$
`while (x < 100) {`	$[1;3]$
`   x := x + 1;`	$[1;2]$
`}`	$\bot$

Example: Interval analysis without widening

Program statement	Approx. state (on entry)
`x := 1;`	$[0; 0]$
`while (x < 100) {`	$[1; 3]$
`  x := x + 1;`	$[1; 3]$
`}`	$\bot$

Example: Interval analysis without widening

Program statement	Approx. state (on entry)
`x := 1;`	$[0; 0]$
`while (x < 100) {`	$[1; 99]$
`  x := x + 1;`	$[1; 98]$
`}`	$\bot$

Jumping forward in time…

Example: Interval analysis without widening

Program statement	Approx. state (on entry)
`x := 1;`	$[0; 0]$
`while (x < 100) {`	$[1; 99]$
`  x := x + 1;`	$[1; 99]$
`}`	$\bot$

Jumping forward in time. . .

Example: Interval analysis without widening

Program statement	Approx. state (on entry)
`x := 1;`	$[0;0]$
`while (x < 100) {`	$[1;100]$
`  x := x + 1;`	$[1;99]$
`}`	$\bot$

Example: Interval analysis without widening

Program statement	Approx. state (on entry)
`x := 1;`	$[0; 0]$
`while (x < 100) {`	$[1; 100]$
`  x := x + 1;`	$[1; 99]$
`}`	$[100; 100]$

Example: Interval analysis without widening

Program statement	Approx. state (on entry)
`x := 1;`	$[0; 0]$
`while (x < 100) {`	$[1; 100]$
`   x := x + 1;`	$[1; 99]$
`}`	$[100; 100]$

Fixed point

Example: Interval analysis without widening

Program statement	Approx. state (on entry)
`x := 1;`	$[0; 0]$
`while (x < 100) {`	$[1; 100]$
`  x := x + 1;`	$[1; 99]$
`}`	$[100; 100]$

Fixed point

In general, we're not guaranteed to reach a fixed point in a finite number of steps (read: impl. may not halt)

Widening

We compute instead the limit of the sequence:

$$X_0 = \bot$$
$$X_{i+1} = X_i \triangledown F^\sharp(X_i)$$

where $\triangledown$ denotes the *widening operator*: an operator with the following properties:

- For all $x, y : x \sqsubseteq (x \triangledown y) \;\wedge\; y \sqsubseteq (x \triangledown y)$
- For any increasing chain $Y_0 \sqsubseteq Y_1 \sqsubseteq Y_2 \sqsubseteq \ldots$ the alternative chain defined as $Y'_0 = Y_0$ and $Y'_{i+1} = Y'_i \triangledown Y_{i+1}$ stabilizes after a finite amount of steps.

Example: Interval analysis with widening

Program statement	Approx. state (on entry)
`x := 1;`	$\bot$
`while (x < 100) {`	$\bot$
`  x := x + 1;`	$\bot$
`}`	$\bot$

Example: Interval analysis with widening

Program statement	Approx. state (on entry)
`x := 1;`	$[0; 0]$
`while (x < 100) {`	$\bot$
`   x := x + 1;`	$\bot$
`}`	$\bot$

Example: Interval analysis with widening

Program statement	Approx. state (on entry)
`x := 1;`	$[0; 0]$
`while (x < 100) {`	$[1; 1]$
`   x := x + 1;`	$\bot$
`}`	$\bot$

Example: Interval analysis with widening

Program statement	Approx. state (on entry)
`x := 1;`	$[0; 0]$
`while (x < 100) {`	$[1; 1]$
`  x := x + 1;`	$[1; 1]$
`}`	$\bot$

Example: Interval analysis with widening

Program statement	Approx. state (on entry)
`x := 1;`	$[0; 0]$
`while (x < 100) {`	$[1; 1] \nabla [1; 2] = [1; +\infty]$
`  x := x + 1;`	$[1; 1]$
`}`	$\bot$

Example: Interval analysis with widening

Program statement	Approx. state (on entry)
`x := 1;`	$[0; 0]$
`while (x < 100) {`	$[1; 1] \nabla [1; 2] = [1; +\infty]$
`  x := x + 1;`	$[1; 99]$
`}`	$[100; +\infty]$

Example: Interval analysis with widening

Program statement	Approx. state (on entry)
`x := 1;`	$[0; 0]$
`while (x < 100) {`	$[1; +\infty] \nabla [1; 100] = [1; +\infty]$
`  x := x + 1;`	$[1; 99]$
`}`	$[100; +\infty]$

Example: Interval analysis with widening

Program statement	Approx. state (on entry)
`x := 1;`	$[0; 0]$
`while (x < 100) {`	$[1; +\infty] \nabla [1; 100] = [1; +\infty]$
`  x := x + 1;`	$[1; 99]$
`}`	$[100; +\infty]$

Stabilized

Example: Interval analysis with widening

Program statement	Approx. state (on entry)
`x := 1;`	$[0;0] \sqsupseteq [0;0]$
`while (x < 100) {`	$[1;+\infty] \sqsupseteq [1;100]$
`   x := x + 1;`	$[1;99] \sqsupseteq [1;99]$
`}`	$[100;+\infty] \sqsupseteq [100;100]$

Stabilized (but we overshot the fixed point)

Thanks to widening, we stabilize in a finite number of steps (read: we always halt)

Narrowing (improved overshooting)

We can compute the limit of the sequence:

$$X_0 = \lim_i Y_i$$
$$X_{i+1} = X_i \vartriangle F^\sharp(X_i)$$

where $\vartriangle$ denotes the *narrowing operator*: an operator with the following properties:

- For all $x, y : (x \vartriangle y) \sqsubseteq x$
- For all $x, y, z : (x \sqsubseteq y \ \wedge \ x \sqsubseteq z) \implies x \sqsubseteq (y \vartriangle z)$
- For any chain Y_i the alternative chain defined as $Y'_0 = Y_0$ and $Y'_{i+1} = Y'_i \vartriangle Y_{i+1}$ stabilizes after a finite amount of steps.

Example: Narrowing our interval analysis

Program statement	Approx. state (on entry)
`x := 1;`	$[0; 0]$
`while (x < 100) {`	$[1; +\infty]$
`  x := x + 1;`	$[1; 99]$
`}`	$[100; +\infty]$

Starting from the overshot fixed point. . .

Example: Narrowing our interval analysis

Program statement	Approx. state (on entry)
`x := 1;`	$[0; 0]$
`while (x < 100) {`	$[1; +\infty] \vartriangle [1; 100] = [1; 100]$
`  x := x + 1;`	$[1; 99]$
`}`	$[100; +\infty]$

Starting from the overshot fixed point. . .

Example: Narrowing our interval analysis

Program statement	Approx. state (on entry)
`x := 1;`	$[0; 0]$
`while (x < 100) {`	$[1; +\infty] \triangle [1; 100] = [1; 100]$
`  x := x + 1;`	$[1; 99]$
`}`	$[100; 100]$

Example: Narrowing our interval analysis

Program statement	Approx. state (on entry)
`x := 1;`	$[0; 0]$
`while (x < 100) {`	$[1; +\infty] \triangle [1; 100] = [1; 100]$
`  x := x + 1;`	$[1; 99]$
`}`	$[100; 100]$

Stabilized

Example: Narrowing our interval analysis

Program statement	Approx. state (on entry)
`x := 1;`	$[0; 0]$
`while (x < 100) {`	$[1; +\infty] \triangle [1; 100] = [1; 100]$
`  x := x + 1;`	$[1; 99]$
`}`	$[100; 100]$

Stabilized (and we even found the fixed point!)

Example: Narrowing our interval analysis

Program statement	Approx. state (on entry)
`x := 1;`	$[0;0]$
`while (x < 100) {`	$[1;+\infty] \vartriangle [1;100] = [1;100]$
`  x := x + 1;`	$[1;99]$
`}`	$[100;100]$

Stabilized (and we even found the fixed point!)

In general, narrowing will stabilize in a finite number of steps on a sound result (may not be the fixed point)

Some words on functional programming and OCaml

Why FP and OCaml?

We’ll use a functional programming language to implement these constructs.

Why FP and OCaml?

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Why?

Why FP and OCaml?

We'll use a functional programming language to implement these constructs.

Why?

→ It's a good fit for the job

- □ Algebraic datatypes and pattern matching are great for this kind of language processing
- □ Microsoft's static device driver verifier is written in OCaml
- □ ASTREÉ is written in OCaml

You are welcome to use Scala, Haskell, SML, F#, . . . if you prefer.

OCaml is an ML dialect

Hence it

- is expression-based, hence everything has a value
- is strongly typed
- is statically scoped
- has algebraic datatypes, lists, tuples, and pattern matching
- has higher-order functions
- ...

In addition it includes some object-oriented extensions (hence the O in OCaml).

Compilers and IDEs

There is both

- a bytecode compiler (`ocamlc`) and
- an optimizing native code compiler (`ocamlopt`)
- a compiler to JavaScript (`js_of_ocaml`)

IDE-wise, for

- emacs I recommend tuareg-mode
- IntelliJ: you tell me!
- Eclipse people recommend: OCaIDE
  `http://www.algo-prog.info/ocaide/`
  `http://www.cs.jhu.edu/~scott/pl/caml/ocaide.shtml`
- VIM: OMLet
- _: please let me know of your findings

OCaml very briefly (1/2)

You bind values to names using `let`:

```
let a = 42
let b = "a string"
let c = (a,b,"third tuple elem")
let d = ["a";"string";"list"]
```

You also use `let` to declare functions:

```
let double x = x + x
```

Catch 0: function application binds stronger than addition: Hence `f x+1` parses as `(f x)+1`

Catch 1: recursive functions must be marked ’`rec`’:

```
let rec fac n = match n with
  | 0 -> 1
  | n -> n * fac (n - 1)
```

OCaml very briefly (2/2)

The `let` token is also used for local declarations (`[]` is nil, `::` is cons):

```
let concat xs ys =
    let rec walk xs = match xs with
        | [] -> ys
        | x::xs' -> x::(walk xs')
    in
    walk xs
```

however without an `end` to finish the block.

Note how OCaml uses `match ... with` to discriminate (pattern match) on a value.

Exercise: write in OCaml a function `sumlist` of type

```
sumlist : int list -> int
```

Catches and Gotchas

Tuples (and pairs) can be written without parens!

Catch 2: Semicolon `';'` separates list elements (rather than comma `','`). For example, compare the types of `[1,2,3]` and `[1;2;3]`

Catch 3: Algebraic datatypes lets us build new datatypes as sums and products:

```
type 'a tree = Leaf of 'a
             | Node of 'a tree * 'a tree
```

However the constructors must be capitalized otherwise it's a parse error!

Catch 4: The evaluation order is unspecified — however the compiler uses right-to-left in practice(!)

OCaml modules

OCaml has a powerful module system with

- signatures (think interface) and
- functors (think `module -> module` function)

Example:

```
module Intset =
  Set.Make (struct
              type t = ...    (* element type *)
              let compare = ...
                        (* element comparison *)

            end)
```

OCaml modules

OCaml has a powerful module system with

- signatures (think interface) and
- functors (think `module -> module` function)

Example:

```
module Intset =
  Set.Make (struct
              type t = int
              let compare n1 n2 =
                if n1 = n2 then 0 else
                  if n1 > n2 then 1 else -1
            end)
```

OCaml modules

OCaml has a powerful module system with

- signatures (think interface) and
- functors (think `module -> module` function)

Example:

```
module Intset =
  Set.Make (struct
              type t = int
              let compare n1 n2 =
                if n1 = n2 then 0 else
                  if n1 > n2 then 1 else -1
            end)
```

Builtin maps are similar:

```
module Mymap = Map.Make(struct ... end)
```

OCaml modules and separate compilation

We can separate the implementation and the interface of a module into two separate files `x.ml` and `x.mli`.

This is equivalent to

```
module X: sig (* contents of file x.mli *) end
        = struct (* contents of file x.ml *) end
```

Catch 5: Files are lower-case, but their module names are capitalized. Hence, the module in file `set.ml` is referred to as `Set`.

If we write

```
module S = struct let f = ... end
```

in a file `foo.ml` then we (need to) refer to `f` as `Foo.S.f`

Relevant links

- Tutorial and toplevel in your browser

  `http://try.ocamlpro.com/`
- A nice OCaml community site with lots of info:

  `http://ocaml.org/`
- OCaml reference manual

  `http://caml.inria.fr/pub/docs/manual-ocaml/`
- Standard library documentation

  `http://caml.inria.fr/pub/docs/manual-ocaml/libref/`
- Jason Hickey's online book

  `http://files.metaprl.org/doc/ocaml-book.pdf`
- Two mailing lists (beginner + main list)
- ...

Let's code something!

Let's implement

- a transition system interface,
- an instantiation thereof, and
- the transition function from the reachable states collecting semantics

Summary

Summary

We have covered

- The what and the how of the course
- The basics of abstract interpretation (transition systems, reachable states collecting semantics, Galois connections, . . .)
- A crash course in OCaml